Concatenated beam splitters, optical feed-forward and the nonlinear sign gate

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We consider a nonlinear sign gate implemented using a sequence of two beam splitters, and consider the use of further sequences of beam splitters to implement feed-forward so as to correct an error resulting from the first beam splitter. We obtain similar results to Scheel *et al.* [Scheel *et al.*, Phys. Rev. A **73**, 034301 (2006)], in that we also find that our feed-forward procedure is only able to produce a very minor improvement in the success probability of the original gate.

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The use of measurement, coupled with linear optical elements can produce, albeit probabilistically, effective optical nonlinearities, a fact which was first realized in the paper of Knill, Laflamme and Milburn [1]. This is of considerable interest because, as demonstrated in this work, such nonlinearities can be used to construct quantum gates, which, even though they succeed with a less that unity probability can, at least in principle, be used for reliable quantum computing [2]. An example of such a gate is the nonlinear sign gate, which transforms an input state $\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$ by flipping the sign of γ [1].

Such optical gates are implemented by sending a single optical mode successively through a sequence of linear optical elements, where the mode may interact with auxiliary modes at one or more beam splitters. The auxiliary modes are measured (using photocounters), and if the measurement results are right, the correct nonlinear transformation will be implemented on the input mode. While in most of these schemes the optical elements are independent of the measurement results, it is possible to chose successive optical elements based on the results of measurements performed on modes that have interacted with the input mode in preceding elements. Such a procedure is an example of feedback [3], which in this case is referred to as *feed- forward*, because the change which is implemented as a result of the measurement is implemented "downstream" at a later point in the sequence.

Here we investigate the use of feed-forward in the implementation of the nonlinear sign (NS) gate. In the absence of feed-forward, it has been shown that the maximum probability with which the gate can be implemented is 1/4 [4, 5], and further that 1/2 is an absolute upper bound [6]. The question is therefore whether feed-forward can be used to significantly increase the probability above 1/4. An investigation of this question has already been made by Scheel *et al.* [7]. Their approach was first to implement the gate, and if the measurement results were incorrect, to feed the output into a further linear optical measurement sequence to correct the transformation. Here we consider an alternative implementation of the NS gate which breaks down into two sequential steps, each involving a beam splitter and a measurement on the associated auxiliary mode [8, 9]. This allows us to modify the elements halfway through the operation of the gate if the measurement at the first beam splitter was not the required one.

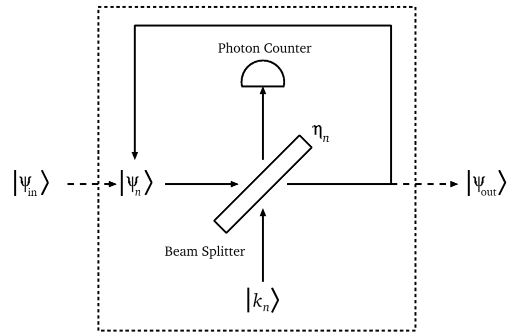

FIG. 1: An optical "feedback" circuit showing one method of implementing the "feed-forward" circuits investigated in the text. In this configuration, the beam is repeatedly sent through a beam splitter whose transmitivity is adjusted at each pass, until the required transformation of the input is obtained at the output mode.

Since a sequence of two beam splitters implements a transformation containing two free parameters, one can expect to be able to perform a correction using this configuration. Here we will restrict ourselves to correction circuits of this form. The resulting complete gate, including feed-forward, will therefore consist of a sequence of concatenated beam splitters, where the number used depends upon the sequence of measurement results. We note that such a configuration could alternatively be implemented with a single controllable beam splitter: when the beam passes through the beam splitter a measurement is made on the auxiliary mode. Depending on the result, the beam is either routed back through the beam splitter for further transformation, or output if the transformation has been completed. This feedback procedure is illustrated in figure 1.

Scheel *et al.* found that they were only able to obtain a very minor improvement in the overall success probability of the gate, and concluded that feed-forward did not appear likely to be a useful tool in the generation of optical nonlinearities. We also report here a negative result; we find that the feed-forward procedure described above is only able to achieve a small increase in the success probability of the initial two-stage gate from which it builds.

To begin let us describe the operation of the two-stage gate devised by Ralph *et al.* [8] and Rudolph and Pan [9] (Note that while we consider standard beam splitter in the following analysis, this gate can also be implemented using polarization beam splitters, which are likely to be easier in practice). This involves mixing the input beam with an auxiliary mode at a single beam splitter, measuring the auxiliary mode, and then repeating the process at a second beam splitter with a second auxiliary mode. This configuration is depicted in Figure 2. One injects a single photon state into the first auxiliary mode, and a vacuum state into the second. The correct transformation is obtained if the input photon is detected in the first auxiliary mode output, and zero photons are detected in the second output. The reason that this works is because each beam splitter and measurement combination performs a nonlinear transformation on the input state. If we send in k photons in the auxiliary mode of the beam splitter, and detect all of them at the auxiliary output, then no photons have been added to the input mode. In this case, the coefficients of the input state transform as

$$\alpha' = \alpha\sqrt{\eta}^{k} \quad (1)$$

$$\beta' = \beta\sqrt{\eta}^{k+1}\left[1-k\xi\right] \quad (2)$$

$$\gamma' = \gamma\sqrt{\eta}^{k+2}\left[1-2k\xi+k(k-1)\xi^2\right], \quad (3)$$

where $\eta \in [0,1]$ is the beam splitter transmitivity, and we have defined $\xi = (1-\eta)/\eta$. The overall norm given by $\mathcal{N} = |\alpha'|^2 + |\beta'|^2 + |\gamma'|^2$ gives the probability that this transformation occurs (i.e. the probability that k photons are detected at the output). If we fix k, then their is one free parameter in the transformation, being the beam splitter transmitivity η.

Note that since the output state must be normalized by dividing by $\mathcal{N}$, there are only two independent parameters in the state (so long as the coefficients α, β and γ remain real). Thus if we wish to perform any desired transformation on the input state that preserves the reality of the coefficients, we need a transformation that has at least two free parameters that can be independently chosen. This is why two successive beam splitters are required to implement the NS gate, since each beam slitter provides a transformation with one free parameter. To perform the gate, we need to chose both parameters, η_1 and η_2, so that the end result is merely a sign change for γ (up to an overall scaling factor). Note, however, that the fact we have two free parameters does not guarantee we can find the transformation we need. For example, the transformation obtained with $k=0$ re-weights the relative magnitudes of α, β, and γ, but only with positive factors. As a result no number of beam splitters with $k=0$ can generate the NS gate. This is why the gate requires one of the beam splitters to have $k=1$.

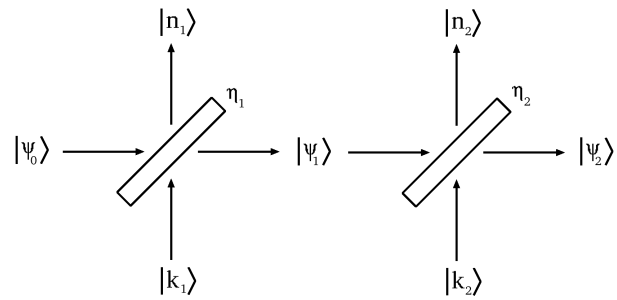

FIG. 2: A sequence of two beam splitters, each accompanied by a measurement on their auxiliary output modes. Such a configuration performs a nonlinear transformation on the input state $|\psi\rangle$ with some probability P. The NS gate of Ralph et al. and Rudolph and Pan has $k_1 = n_1 = 1$, $k_2 = n_2 = 0$ and a success probability of $P = 0.2265$. The NS gate can also be implemented with $k_1 = n_1 = k_2 = n_2 = 1$, giving a success probability of $P = 0.209$.

The set of transformations displayed in Eqs.(3) does not, however, exhaust those which are possible with a beam splitter followed by a measurement on the auxiliary output. Such an element will also add or subtract n photons from the mode if we detect n fewer or n more photons at the auxiliary output than we injected into the auxiliary input. Doing so also generates a new set of transformations, as well as changing the photon number. If we inject k photons, and detect $k-1$ photons, then for $k \geq 2$ the transformation becomes

$$\alpha' = \alpha\sqrt{k(k-1)(1-\eta)}\sqrt{\eta}^{k-1} \quad (4)$$

$$\beta' = \beta\sqrt{2k(k-1)(1-\eta)}\sqrt{\eta}^{k}\left[1-(k-1)\xi/2\right] \quad (5)$$

$$\gamma' = \gamma\sqrt{(3/2)k(k-1)(1-\eta)}\sqrt{\eta}^{k+1} \times\left[1-(k-1)\xi+(k-1)(k-2)\xi^2/3\right], \quad (6)$$

where we have removed an overall phase factor of $-i$, and where the unnormalized output state is now $\alpha'|1\rangle + \beta'|2\rangle + \gamma'|3\rangle$, since a photon has been added to the mode. For $k=1$ the transformation is $\alpha' = \alpha\sqrt{1-\eta}$, $\beta' = \beta\sqrt{2\eta(1-\eta)}$ and $\gamma' = \gamma\eta\sqrt{3(1-\eta)/2}$. If, on the other hand, we detect one more photon than we inject, then we will remove n photons from the input mode. This will remove from the output state the subspace that has fewer than n photons. Thus, if the input state has a component $\alpha|0\rangle$, and we remove one photon from the mode, this component will be removed, so that α will be set to zero, and the information contained in the value of α will be lost. As far as the operation of a gate is concerned, this results in a failure that cannot be corrected downstream using feed-forward. However, if we have already added n photons to the input mode, then we can remove them without losing any of the coefficients. If we start with an input state $\alpha'|1\rangle + \beta'|2\rangle + \gamma'|3\rangle$, inject k photons at the auxiliary port, and remove a photon from the mode, the resulting transformation for $k > 0$ is

$$\alpha' = \alpha\sqrt{(1-\eta)(k+1)/k}\sqrt{\eta}^k \quad (7)$$

$$\beta' = \beta\sqrt{2(1-\eta)(k+1)/k}\sqrt{\eta}^{k+1}\left[1-k\xi/2\right] \quad (8)$$

$$\gamma' = \gamma\sqrt{6(1-\eta)(k+1)/k}\sqrt{\eta}^{k+2} \times \left[1-k\xi+k(k-1)\xi^2/6\right]. \quad (9)$$

where we have omitted an overall phase factor of $-i$. For $k = 0$ the transformation is $\alpha' = \alpha\sqrt{1-\eta}$, $\beta' = \beta\sqrt{2\eta(1-\eta)}$ and $\gamma' = \gamma\sqrt{6(1-\eta)}\eta$.

Finally, it is important to note that the transformation that is performed on the input coefficients depends not only upon the number of injected and detected photons, but also on the number of photons in the input mode (that is, on the number of photons which has previously been added to the input beam). If we both inject and detect k photons, but have an input state with one extra photon (that is, $|\psi_{\rm in}\rangle = \alpha|1\rangle + \beta|2\rangle + \gamma|3\rangle$), then the transformation is

$$\alpha' = \alpha\sqrt{\eta}^{k+1}\left[1-k\xi\right] \quad (10)$$

$$\beta' = \beta\sqrt{\eta}^{k+2}\left[1-2k\xi+k(k-1)\xi^2\right] \quad (11)$$

$$\gamma' = \gamma\sqrt{\eta}^{k+3}\left[1-3k\xi+3k(k-1)\xi^2-\frac{k!}{(k-3)!}\xi^3\right],$$

Before we consider the use of feed-forward, we investigate whether it is possible to construct an NS gate with two beam splitters by using different numbers of input photons than those in the gate described above. We do this because different options for implementing the gate might provide alternative options for correcting the output of the gate using feed-forward.

It will be useful to introduce a compact notation to describe a given sequence of beam splitters. We will denote a beam splitter with k photons injected into the auxiliary mode, and n photons detected at the auxiliary output by the pair (k, n). A sequence of two or more beam splitters will then be denoted $[(k_1, n_1), (k_2, n_2), \ldots]$. In this notation, the NS gate of Ralph *et al.* is $[(1, 1), (0, 0)]$. Note that the configuration $[(n, n), (m, m)]$ is completely equivalent to $[(m, m), (n, n)]$, since no photons are added to the beam by either element. We now explore these configurations for $n, m = 0 \ldots 4$. We find that $[(1, 1), (m, m)]$ can be used to produce an NS gate, however, the success probability slowly falls as m is increased. In particular, $[(1, 1), (1, 1)]$ will generate an NS gate with success probability $P = 0.209$, a little below that of the [(1,1),(0,0)] configuration. (This requires the transmitivities $\eta_1 = 0.2275$ and $\eta_2 = 0.91968$, or vice versa.) Conversely, configurations of the form $[(n, n), (0, 0)]$ will only generate an NS gate for $n = 1$. Configurations in which both n and m are larger than 1 will generate NS gates but with significantly lower success probabilities. We also note that the configuration $[(1, 0), (0, 1)]$, which injects a photon into the mode at the first beam splitter and removes it at the second beam splitter, cannot generate an NS gate because all the transformation factors for the coefficients are positive. Thus, as one increases the number of auxiliary photons used, the success probability of the resulting NS gates get *worse* rather than better, thus the scheme devised by Ralph *et al.* and Rudolph and Pan has what appears to be the highest success probability using only two concatenated beam splitters. [10]

TABLE I: Success probabilities for three concatenated beam splitters

Sequence	η_1	η_2	η_3	P
(1,0),(0,1),(1,1)	0.9197	0.2947	0.2567	0.0145
(1,0),(1,2),(0,0)	0.9197	0.1472	0.5137	0.0202
(1,0),(1,2),(1,1)	0.9197	0.1511	0.8398	0.0173
(1,0),(0,0),(1,2)	0.9197	0.5137	0.1472	0.0104
(1,0),(1,1),(0,1)	0.9197	0.6500	0.4182	0.0042
(1,0),(1,1),(1,2)	0.2265	0.3315	0.0531	0.0088
”	0.9197	0.8690	0.1766	0.0127

We now consider the use of feed-forward to increase the success probability of a two-element NS gate. To implement this feed-forward one first measures the auxiliary output of the first beam splitter. If this output is correct for the implementation of the gate, then the second beam-splitter is implemented as usual. However, if the output is not correct, then we route the output beam to a new set of two beam splitters designed to transform this output so as to produce the correct transformation for the NS gate. We expect that only two beam splitters will be required to do this, since, as mentioned above, this provides a transformation with two tunable parameters.

To implement feed-forward we chose a version of the NS gate which has one photon injected into the first beam splitter. In this case, if we measure no photons in the auxiliary output, we have an error which is potentially correctable. As discussed above, an NS gate can be constructed in this manner by choosing $\eta_1 = 0.2265$ (using the configuration [(1,1),(0,0)]) or $\eta_1 = 0.9197$ or 0.2275 (using the configuration [(1,1),(1,1)]). Since the last of these is not significantly different from the first, we need consider only the first two. We will therefore apply various possible correction circuits to an error with these two values of η_1 to determine which provides the highest probability of a successful correction.

There are a variety of two-element configurations which can be used for correction. The only requirement is that the sequence subtract a photon from the input beam, since the detection of zero photons at the first beam-splitter has added a photon to the beam. We will examine sequences which will do this using low numbers of auxiliary photons. We investigate the sequences [(0,1),(0,0)], [(0,1),(1,1)], [(1,2),(0,0)] and [(1,2),(1,1)] in which the photon is extracted at the first beam splitter, and the sequences [(0,0),(0,1)], [(1,1),(0,1)], [(0,0),(1,2)], and [(1,1),(1,2)] in which the photon is extracted at the second beam splitter. To calculate the success probably for the correction, we consider the sequence of three beam splitters, where the first is the initial beam splitter at which the error $(1, 0)$ occurs, and the second two are the pair used for correction. The transmitivity of the initial beam splitter is set at one of the three possible values given above. We then find all solutions for the two transmitivities of the second pair which generate the correct NS gate output, and from these obtain the success probabilities. We present the success probabilities for the various beam splitter triples in table I. We have only included those that provide a solution with an appreciable success probability.

Examining the table, note first that only one of the sequences is able to correct for error when $\eta_1 = 0.2265$. That is, the gate that uses an auxiliary photon at both inputs has more ways to perform a correction, and these give higher success probabilities. However, this gate has itself a reduced success probability over the gate with $\eta_1 = 0.2265$. This is similar to the behavior noted by Scheel *et al.*, that if one wants to increase the probability of a successful correction, then one must reduce the success probability of the initial gate.

The highest success probability for correction is provided by the sequence [(1,0),(1,2),(0,0)], and this is only 0.0202. Our feed-forward procedure is therefore only able to increase the success probability of the initial gate a by small amount. One could consider further corrections, in which an error at the second beam splitter could be corrected with a further beam splitter pair. However, such a correction would involve a sequence of four beam splitters, and we have seen that moving from two beam splitters to three causes a big reduction in the success probability. We can therefore expect a sequence of four beam splitters to produce only a miniscule increase in the total success probability. It appears therefore that feed-forward, certainly in the configuration we have considered, is not likely to provide a means to significantly improve the success probability of linear optical quantum gates.

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